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Hom-Lie algebra structures on semi-simple Lie algebras ☆

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Abstract

Hom-Lie algebras can be considered as a deformation of Lie algebras. In this note, we prove that the hom-Lie algebra structures on finite-dimensional simple Lie algebras are trivial. We find when a finite-dimensional semi-simple Lie algebra admits non-trivial hom-Lie algebra structures and the isomorphic classes of non-trivial hom-Lie algebras are determined.

Keywords: Hom-Lie algebra; Lie algebra

1. Introduction and preliminaries

Jonas T. Hartwig, Daniel Larsson, Sergei D. Silvestrov developed a new approach to the deformation theory of Witt and Virasoro algebras using σ-derivations in their paper [1]. They also introduced the concept of a hom-Lie algebra, which is a non-associative algebra satisfying the skew symmetry and the σ-twisted Jacobi identity. When $\sigma = id$, the hom-Lie algebras degenerate to exactly the Lie algebras. In this note, we are interested in the following problem. Does there exist any non-trivial hom-Lie algebra structures on the complex finite-dimensional semi-simple Lie algebras? And if yes, what are these hom-Lie algebras and when they are isomorphic? In Section 2 we prove that simple Lie algebras do not admit any non-trivial hom-Lie algebra structures. In Section 3, we prove that a finite-dimensional semi-simple Lie algebra L admits non-trivial

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hom-Lie algebra structures if and only if there are some isomorphic simple ones among the simple ideals of L, and the non-trivial hom-Lie algebra structures are determined.

The Scalar field $\mathbb{K}$ for algebras considered in this paper is of characteristic zero. First let us recall some notions in [1].

Definition 1.1. A hom-Lie algebra (L, σ) is a non-associative algebra L together with an algebra homomorphism $\sigma : L \to L$, such that

(HL1) $[x, y] = -[y, x]$,
(HL2) $[(id + \sigma)(x), [y, z]] + [(id + \sigma)(y), [z, x]] + [(id + \sigma)(z), [x, y]] = 0$

for all $x, y, z \in L$, where $[\,,\,]$ denotes the product in L.

We call Axiom (HL1) the skew symmetry and Axiom (HL2) the σ-twisted Jacobi identity.

Remark 1. Taking $\sigma = id$ in the definition above gives us the definition of a Lie algebra. Hence hom-Lie algebras include Lie algebras as a subclass.

Remark 2. For any vector space V, if we put

$$[x, y] = 0$$

for any $x, y \in V$, then (V, σ) is obviously a hom-Lie algebra for any linear map $\sigma : V \to V$, since the conditions are trivially satisfied. We call these algebras abelian hom-Lie algebras.

Definition 1.2. A homomorphism (respectively isomorphism) of hom-Lie algebras $\varphi : (L_1, \sigma_1) \to (L_2, \sigma_2)$ is an algebra homomorphism (respectively isomorphism) from L_1 to L_2 such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$. In other words, the following diagram

$$\begin{tikzcd} L_1 \arrow[r, "\varphi"] \arrow[d, "\sigma_1"] & L_2 \arrow[d, "\sigma_2"] \\ L_1 \arrow[r, "\varphi"] & L_2 \end{tikzcd}$$

commutes.

For two hom-Lie algebras (L_1, σ_1) and (L_2, σ_2), as in the Lie algebra case, we can define a hom-Lie algebra structure on the space $L_1 \oplus L_2$ by defining $[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + [x_2, y_2]$ and $\sigma_1 \oplus \sigma_2(x_1 + x_2) = \sigma_1(x_1) + \sigma_2(x_2)$ for $x_1, y_1 \in L_1, x_2, y_2 \in L_2$. We call this hom-Lie algebra *the direct sum of* (L_1, σ_1) *and* (L_2, σ_2) and denote it by $(L_1 \oplus L_2, \sigma_1 \oplus \sigma_2)$.

2. Hom-Lie algebras on simple Lie algebras

In this section, we prove that hom-Lie algebra structures on finite-dimensional simple Lie algebras are trivial.

For any simple Lie algebra L, a homomorphism $\sigma : L \to L$ is either trivial (i.e., $\sigma = 0$) or an automorphism of the Lie algebra L. If $\sigma = 0$, then the hom-Lie algebra (L, σ) is the Lie algebra L itself. Therefore, we suppose that σ is an automorphism of L in the sequel.

Suppose (L, σ) and (L, τ) are two hom-Lie algebra structures on the simple Lie algebra L, where σ, τ are Lie algebra automorphisms of L. If (L, σ) and (L, τ) are isomorphic as hom-Lie algebras, then there exist a Lie algebra automorphism $\varphi : L \to L$ such that the following diagram

$$\begin{tikzcd} L_1 \arrow[r, "\varphi"] \arrow[d, "\sigma"] & L_2 \arrow[d, "\tau"] \\ L_1 \arrow[r, "\varphi"] & L_2 \end{tikzcd}$$

commutes. That is $\varphi \circ \sigma = \tau \circ \varphi$. Therefore $\sigma = \varphi^{-1} \circ \tau \circ \varphi$. So we get

Proposition 2.1. *Let L be a simple Lie algebra, then two hom-Lie algebra structures (L, σ) and (L, τ) are isomorphic if and only if the two Lie algebra automorphisms σ and τ are conjugate.*

By Proposition 2.1, we only need to determine the hom-Lie algebra structures (L, σ) on simple Lie algebras L with a Lie algebra automorphisms σ.

Proposition 2.2. *Let L be a finite-dimensional simple Lie algebra. If (L, σ) is a hom-Lie algebra, then $\sigma = id$. In other words, hom-Lie algebra structures on simple Lie algebras are trivial.*

Proof. From [2], we can assume that the automorphism σ leaves the Cartan subalgebra $\mathcal{H}$ of L stable.

By Proposition 8.1 in [3], an inner automorphism σ of L can be written as $\sigma = e^{\operatorname{ad} h}$ for some $h \in \mathcal{H}$ and an outer automorphism of L can be expressed as $\sigma = \rho e^{\operatorname{ad} h}$ for some $h \in \mathcal{H}$ and a diagram automorphism ρ of L, which implies that simple Lie algebras of type other than A_l $(l \geqslant 2)$, D_l $(l \geqslant 4)$ and E_6 have no outer automorphisms. It is clear that $e^{\operatorname{ad} h}$ leaves $\mathcal{H}$ point-wise fixed, and $e^{\operatorname{ad} h}(x_\alpha) = e^{\alpha(h)} x_\alpha$ for all root vectors $x_\alpha \in L_\alpha$, where L_α is the root space of root $\alpha \in \Delta$.

We will proceed our proof according to the rank of simple Lie algebras.

For the rank one Lie algebra $sl(2, \mathbb{C})$, we can choose h, e, f as its standard basis such that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Since $sl(2, \mathbb{C})$ has only inner automorphisms whose conjugate class can be expressed as $\sigma = e^{\operatorname{ad}(kh)}$ for some $k \in \mathbb{C}$, we have

$$\begin{aligned} \big[\sigma(h), [e, f]\big] &= 0, \\ \big[\sigma(e), [f, h]\big] &= e^{2k}[e, 2f] = 2e^{2k} h, \\ \big[\sigma(f), [h, e]\big] &= e^{-2k}[f, 2e] = -2e^{-2k} h. \end{aligned}$$

Therefore the deformed Jacobi identity holds if and only if $k = 0$, or equivalently $\sigma = id$.

Now let L be a simple Lie algebra whose rank is at least 2 and $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ ($\ell \geqslant 2$) be the simple roots for the root system Δ of L. Since L is a Lie algebra, the σ-deformed Jacobi identity degenerates to

$$\big[\sigma(x), [y, z]\big] + \big[\sigma(y), [z, x]\big] + \big[\sigma(z), [x, y]\big] = 0 \quad \text{for all } x, y, z \in L.$$

If the σ-deformed Jacobi identity holds, we will have

$$\big[\sigma(\mathcal{H}), [L_{\alpha_i}, L_{\alpha_j}]\big] + \big[\sigma(L_{\alpha_i}), [L_{\alpha_j}, \mathcal{H}]\big] + \big[\sigma(L_{\alpha_j}), [\mathcal{H}, L_{\alpha_i}]\big] = 0,$$

or equivalently

$$\big[\sigma(\mathcal{H}), [L_{\alpha_i}, L_{\alpha_j}]\big] = \big[\sigma(L_{\alpha_i}), [\mathcal{H}, L_{\alpha_j}]\big] + \big[[\mathcal{H}, L_{\alpha_i}], \sigma(L_{\alpha_j})\big]. \tag{1}$$

Choosing $h' \in \mathcal{H}$ such that $\alpha_i(h') = 0$ and $\alpha_j(h') \neq 0$, since $\sigma(h') = e^{\operatorname{ad} h}(h') = h'$ we have

$$\begin{aligned}
\big[h', [L_{\alpha_i}, L_{\alpha_j}]\big] &= \big[\sigma(L_{\alpha_i}), [h', L_{\alpha_j}]\big] + \big[[h', L_{\alpha_i}], \sigma(L_{\alpha_j})\big] \\
&= \alpha_j(h')\big[e^{\alpha_i(h)} L_{\alpha_i}, L_{\alpha_j}\big] + \alpha_i(h')\big[L_{\alpha_i}, e^{\alpha_j(h)} L_{\alpha_j}\big] \\
&= e^{\alpha_i(h)} \alpha_j(h')[L_{\alpha_i}, L_{\alpha_j}].
\end{aligned} \tag{2}$$

Select α_i, α_j such that $\alpha_i + \alpha_j$ is a positive root, then (2) becomes

$$\alpha_j(h') L_{\alpha_i + \alpha_j} = e^{\alpha_i(h)} \alpha_j(h') L_{\alpha_i + \alpha_j},$$

therefore $e^{\alpha_i(h)} = 1$, and hence $\alpha_i(h) = 0$. Similarly, if we choose $h' \in \mathcal{H}$ in (2) such that $\alpha_i(h') \neq 0$ and $\alpha_j(h') = 0$, we will get $\alpha_j(h) = 0$. Since the Dynkin diagram of a simple Lie algebra is connected, we have $\alpha_i(h) = 0$ for $i = 1, 2, \dots, \ell$ [2]. The fact that $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ constitute a basis of $\mathcal{H}^*$, the dual space of the Cartan subalgebra $\mathcal{H}$, implies that $h = 0$, and so $\sigma = e^{\operatorname{ad} h}$ is the identity map on the whole simple Lie algebra L.

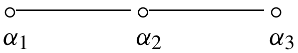
If σ is an outer automorphism of L, then we can assume $\sigma = \rho e^{\operatorname{ad} h}$ for some $h \in \mathcal{H}$, where ρ is a diagram automorphism of L. Then (1) becomes

$$\big[\rho(\mathcal{H}), [L_{\alpha_i}, L_{\alpha_j}]\big] = \big[\rho e^{\operatorname{ad} h}(L_{\alpha_i}), [\mathcal{H}, L_{\alpha_j}]\big] + \big[[\mathcal{H}, L_{\alpha_i}], \rho e^{\operatorname{ad} h}(L_{\alpha_j})\big], \tag{3}$$

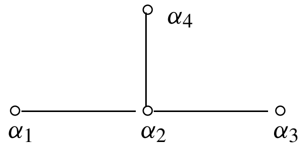
that is

$$\big[\rho(\mathcal{H}), [L_{\alpha_i}, L_{\alpha_j}]\big] = e^{\alpha_i(h)} \alpha_j(\mathcal{H})[L_{\rho(\alpha_i)}, L_{\alpha_j}] + e^{\alpha_j(h)} \alpha_i(\mathcal{H})[L_{\alpha_i}, L_{\rho(\alpha_j)}]. \tag{4}$$

Choosing α_i, α_j such that $\alpha_i + \alpha_j \in \Delta$, the fact $\dim L_\alpha = 1$ for any $\alpha \in \Delta$ shows that (4) holds only when $\alpha_i + \alpha_j = \rho(\alpha_i) + \alpha_j$ or $\alpha_i + \alpha_j = \alpha_i + \rho(\alpha_j)$ holds. In any case, we have that if (4) holds, then ρ fixes at least one of α_i and α_j. From the Dynkin diagrams of simple Lie algebras we know that any such order two diagram automorphism ρ can be restricted as an automorphism to the subalgebra $sl(4, \mathbb{C})$, and the order three diagram automorphism exists only for the Lie algebra D_4 (see p. 58 in [2]). In this note, the numbering of the nodes in the Dynkin diagrams of A_3 and D_4 is as follows



Dynkin diagram of A_3



Dynkin diagram of D_4

The order two diagram automorphism ρ fixes α_2 and exchanges α_1 and α_3. Applying equality (4) to $sl(4,\mathbb{C})$ with $i=1,\ j=2$, we get

$$\big[\rho(h_3),[L_{\alpha_1},L_{\alpha_2}]\big]=e^{\alpha_1(h)}\alpha_2(h_3)[L_{\rho(\alpha_1)},L_{\alpha_2}]+e^{\alpha_2(h)}\alpha_1(h_3)[L_{\alpha_1},L_{\rho(\alpha_2)}]. \tag{5}$$

It is easy to see that both sides of (5) are not zero, but the left-hand side belongs to $L_{\alpha_1+\alpha_2}$ while the right-hand side belongs to $L_{\alpha_2+\alpha_3}$, which is impossible.

The order three diagram automorphism ρ for D_4 fixes α_2 and permutes $\{\alpha_1,\alpha_3,\alpha_4\}$. Applying (4) to D_4 with $i=1,\ j=2$, we get

$$\big[\rho(h_3),[L_{\alpha_1},L_{\alpha_2}]\big]=e^{\alpha_1(h)}\alpha_2(h_3)[L_{\rho(\alpha_1)},L_{\alpha_2}]+e^{\alpha_2(h)}\alpha_1(h_3)[L_{\alpha_1},L_{\rho(\alpha_2)}]. \tag{6}$$

It is easy to see that both sides of (6) are not zero, but the left-hand side belongs to $L_{\alpha_1+\alpha_2}$ while the right-hand side belongs to $L_{\alpha_2+\alpha_3}$, which is also impossible. □

3. Hom-Lie algebras on semi-simple Lie algebras

Let L be a complex finite-dimensional semi-simple Lie algebra. We have the following decomposition of L into direct sum of simple ideals $L_1,L_2,\ldots,L_r$:

$$L=m_1L_1\oplus m_2L_2\oplus\cdots\oplus m_rL_r,$$

where $L_1,L_2,\ldots,L_r$ are non-isomorphic simple ideals of L, and by the notation m_iL_i we mean that L_i is a simple ideal of L with multiplicity m_i ($m_i\geqslant 1$), i.e.,

$$m_iL_i=L_i\oplus L_i\oplus\cdots\oplus L_i.$$

If $\varphi:L\to L$ is a homomorphism of L, then since the kernel of φ is a direct sum of some simple ideals of L, the Lie algebra homomorphism φ can be decomposed as $\varphi=\varphi_1\oplus\varphi_2\oplus\cdots\oplus\varphi_r$, where $\varphi_i=\varphi|_{m_iL_i}$, the restriction of φ to m_iL_i. Because the L_i's are non-isomorphic simple Lie ideals of L and the homomorphic image $\varphi_i(m_iL_i)$ is either zero or contains a simple ideal isomorphic to L_i, we have $\varphi_i(m_iL_i)\subseteq m_iL_i$ for $i=1,2,\ldots,r$. Thus the pair (L,φ) is a hom-Lie algebra if and only if the skew symmetry and the σ-twisted Jacobi identity in Definition 1.1 are satisfied. The skew symmetry is automatically satisfied since L is a Lie algebra. The σ-twisted Jacobi identity is equivalent to

$$\big[(id+\varphi_i)(x),[y,z]\big]+\big[(id+\varphi_i)(y),[z,x]\big]+\big[(id+\varphi_i)(z),[x,y]\big]=0$$

for all $x,y,z\in m_iL_i$ and $1\leqslant i\leqslant r$. That is, (L,φ) is a hom-Lie algebra if and only if each (m_iL_i,φ_i) is a hom-Lie algebra for $i=1,2,\ldots,r$. So we have

Proposition 3.1. *Let $L = m_1L_1 \oplus m_2L_2 \oplus \cdots \oplus m_rL_r$ be a semi-simple Lie algebra with non-isomorphic summands L_i and $\varphi : L \to L$ a homomorphism. Then (L, φ) is a hom-Lie algebra if and only if each (m_iL_i, φ_i) is a hom-Lie algebra for $i = 1, 2, \ldots, r$, where $\varphi_i = \varphi|_{m_iL_i}$. Moreover, as hom-Lie algebras we have the following direct-sum decomposition*

$$(L, \varphi) = (m_1L_1, \varphi_1) \oplus (m_2L_2, \varphi_2) \oplus \cdots \oplus (m_rL_r, \varphi_r).$$

Corollary 3.1. *Let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ be a semi-simple Lie algebra with non-isomorphic summands L_i. Then L does not admit any non-trivial hom-Lie algebra structures.*

Proof. This is a direct result of Propositions 2.2 and 3.1. □

From Proposition 3.1, the hom-Lie algebra structures on semi-simple Lie algebras $L = m_1L_1 \oplus m_2L_2 \oplus \cdots \oplus m_rL_r$ are completely determined by those on the semi-simple summands $m_iL_i = L_i \oplus L_i \oplus \cdots \oplus L_i$. In what follows we will write the semi-simple Lie algebra $mL = L \oplus L \oplus \cdots \oplus L$ with m copies of isomorphic simple ideals L as $mL = (L, L, \ldots, L)$. Elements in mL will be written as $X_{i_1i_2\cdots i_s}(x_1, x_2, \ldots, x_s)$ $(1 \leqslant s \leqslant m)$, which means the i_kth entry is $x_k \in L$ for $k = 1, 2, \ldots, s$, and other entries not written down are all zero. The semi-simplicity of mL implies that any Lie algebra homomorphism $\rho : mL \mapsto mL$ can be decomposed as $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_m$, where ρ_i is the restriction of ρ to the ith summand L. Let P_i be the projection homomorphism from mL into its ith component L, then the composition $\rho_i \circ P_i$ is a Lie algebra homomorphism on mL. In sequel we will regard ρ_i as $\rho_i \circ P_i$.

Lemma 3.1. *The simple ideals of mL are either the "component" simple ideals $L_i = \{X_i(x) \mid x \in L\}$ $(i = 1, 2, \ldots, m)$ or the "diagonal" simple ideals $L_{i_1i_2\cdots i_l} = \{X_{i_1i_2\cdots i_l}(x, \ldots, x) \mid x \in L\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant m$ and $l \geqslant 2$.*

Proof. It is easy to verify that $L_i = \{X_i(x) \mid x \in L\}$ and $L_{i_1i_2\cdots i_l} = \{X_{i_1i_2\cdots i_l}(x, \ldots, x) \mid x \in L\}$ are all simple ideals of mL. Let $S \subseteq mL$ be a simple ideal of mL.

Case 1. S is composed of elements of the form $X_i(x)$. If $X_i(x), X_j(y) \in S$ and $i \neq j$, then since S is a simple ideal of mL, we have that

$$S \supseteq \left[X_i(x), mL\right] = \left\{X_i(x) \,\middle|\, x \in L\right\} = L_i \quad \text{and}$$
$$S \supseteq \left[X_j(y), mL\right] = \left\{X_j(y) \,\middle|\, y \in L\right\} = L_j,$$

which contradicts with our assumption that S is simple. Thus $i = j$, and S is a *component* simple ideal L_i for some i.

Case 2. If $X_{i_1i_2\cdots i_l}(x_1, x_2, \ldots, x_l) \in S$ with $l \geqslant 2$, we must have that $x_1 = x_2 = \cdots = x_l$ (up to scalars). Otherwise, if there are two entries in $X_{i_1i_2\cdots i_l}(x_1, x_2, \ldots, x_l) \in S$ that are not proportional, say $x_1 \neq kx_2$ for any $k \in \mathbb{K}$, then we have

$$S \supseteq \left[X_{i_1i_2\cdots i_l}(x_1, x_2, \ldots, x_l), mL\right] \supseteq \left[X_{i_1i_2\cdots i_l}(x_1, x_2, \ldots, x_l), L_1 \oplus L_2\right] \supseteq L_1 \oplus L_2.$$

Again a contradiction with the simpleness of S. Therefore S is a *diagonal* simple ideal $L_{i_1i_2\cdots i_l} = \{X_{i_1i_2\cdots i_l}(x, \ldots, x) \mid x \in L\}$ for some $l \geqslant 2$ and $1 \leqslant i_1 < i_2 \cdots < i_l \leqslant m$. □

By Lemma 3.1, the image $\mathrm{Im}(\rho_i)$ of a non-trivial homomorphism ρ_i is either a *component* simple ideal $L_i = \{X_i(x) \mid x \in L\}$ $(i = 1, 2, \ldots, m)$ or a *diagonal* simple ideal $L_{i_1 i_2 \cdots i_l} = \{X_{i_1 i_2 \cdots i_l}(x, \ldots, x) \mid x \in L\}$ with $l \geqslant 2$. In fact, the non-trivial ρ_i can be realized as the following two types of homomorphisms.

For an automorphism α of L, we define $\Psi_\alpha(i,j) : L_i \mapsto mL$ by

$$\Psi_\alpha(i,j)\big(X_i(x)\big) = X_j\big(\alpha(x)\big).$$

Lemma 3.2. *If $i \neq j$, then $\Psi_\alpha(i,j)$ is a homomorphism of mL and $(mL, \Psi_\alpha(i,j))$ is a hom-Lie algebra.*

Proof. As explained before, $\Psi_\alpha(i,j)$ is actually $\Psi_\alpha(i,j) \circ P_i$ and thus $\Psi_\alpha(i,j)$ is a vector space homomorphism on mL. It easy to check that

$$\begin{aligned} \Psi_\alpha(i,j)\big[X_i(x), X_i(y)\big] &= \Psi_\alpha(i,j)\big(X_i\big([x,y]\big)\big) \\ &= X_j\big(\alpha\big([x,y]\big)\big) = X_j\big(\big[\alpha(x), \alpha(y)\big]\big) = \big[X_j\big(\alpha(x)\big), X_j\big(\alpha(y)\big)\big] \\ &= \big[\Psi_\alpha(i,j)\big(X_i(x)\big), \Psi_\alpha(i,j)\big(X_i(y)\big)\big]. \end{aligned} \tag{7}$$

Thus $\Psi_\alpha(i,j)$ is indeed a Lie algebra homomorphism of mL. In order to prove that $(mL, \Psi_\alpha(i,j))$ is a hom-Lie algebra, we must verify that the $\Psi_\alpha(i,j)$-twisted Jacobi identity in Definition 1.1 is satisfied. The $\Psi_\alpha(i,j)$-twisted Jacobi identity can be re-written as:

$$\begin{aligned} &\big[X_i(x), \big[X_i(y), X_i(z)\big]\big] + \big[X_i(y), \big[X_i(z), X_i(x)\big]\big] + \big[X_i(z), \big[X_i(x), X_i(y)\big]\big] \\ &\quad + \big[\Psi_\alpha(i,j)\big(X_i(x)\big), \big[X_i(y), X_i(z)\big]\big] + \big[\Psi_\alpha(i,j)\big(X_i(y)\big), \big[X_i(z), X_i(x)\big]\big] \\ &\quad + \big[\Psi_\alpha(i,j)\big(X_i(z)\big), \big[X_i(x), X_i(y)\big]\big] = 0. \end{aligned} \tag{8}$$

The sum of the first three terms is automatically zero because of the Jacobi identity of the Lie algebra L_i while each of the last three terms is obviously zero. For example, we have $[\Psi_\alpha(i,j)(X_i(x)), [X_i(y), X_i(z)]] = 0$ because

$$\begin{aligned} \big[\Psi_\alpha(i,j)\big(X_i(x)\big), \big[X_i(y), X_i(z)\big]\big] &= \big[X_j\big(\alpha(x)\big), \big[X_i(y), X_i(z)\big]\big] \\ &\in \big[L_j, [L_i, L_i]\big] \subseteq [L_j, L_i] = \{0\}. \end{aligned} \tag{9}$$

Therefore $(mL, \Psi_\alpha(i,j))$ is a hom-Lie algebra. □

Remark 3. If $i = j$, then $\Psi_\alpha(i,j)$ is an automorphism of the simple Lie algebra L_i, which is completely determined by the automorphism $\alpha : L \mapsto L$. Equality (8) is just the α-twisted Jacobi identity on the simple Lie algebra $L_i \cong L$. By Proposition 2.2, α is the identity automorphism of L and hence the hom-Lie algebra $(mL, \Psi_\alpha(i,j))$ is trivial. We consider only the homomorphisms $\Psi_\alpha(i,j)$ for $i \neq j$ here because we are interested in the non-trivial hom-Lie algebra structures on mL.

For an automorphism β of L and any subset $\{i_1, i_2, \ldots, i_l\} \subseteq \{1, 2, \ldots, m\}$ we define $\Psi_\beta(i, i_1 i_2 \cdots i_l) : L_i \mapsto mL$ by

$$\Psi_\beta\big(X_i(x)\big) = X_{i_1 i_2 \cdots i_l}\big(\beta(x), \beta(x), \ldots, \beta(x)\big).$$

Lemma 3.3. *If $i \notin \{i_1, i_2, \ldots, i_l\}$, then $\Psi_\beta(i, i_1 i_2 \cdots i_l)$ is a homomorphism of mL and $(mL, \Psi_\beta(i, i_1 i_2 \cdots i_l))$ is a hom-Lie algebra.*

The proof is similar to that of Lemma 3.2.

Remark 4. We consider here only the homomorphisms $\Psi_\beta(i, i_1 i_2 \cdots i_l)$ for $i \notin \{i_1, i_2, \ldots, i_l\}$. As explained in Remark 3, when $i \in \{i_1, i_2, \ldots, i_l\}$, the hom-Lie algebra $(mL, \Psi_\beta(i, i_1 i_2 \cdots i_l))$ will be trivial.

Definition 3.1. Two automorphisms σ and τ of a simple Lie algebra L are said to be quasi-conjugate if there exist automorphisms ϕ and φ of L such that $\phi \circ \sigma = \tau \circ \varphi$, that is, the following diagram

$$\begin{tikzcd} L \arrow[r, "\sigma"] \arrow[d, "\varphi"] & L \arrow[d, "\phi"] \\ L \arrow[r, "\tau"] & L \end{tikzcd}$$

commutes.

Remark 5. In Definition 3.1, if $\varphi = \phi$, then σ and τ are not only quasi-conjugate but also conjugate. Thus quasi-conjugation is a generalization of conjugation.

In following propositions we give a necessary and sufficient condition for the hom-Lie algebras $\Psi_\alpha(i, j)$ and $\Psi_\beta(i, i_1 i_1 \cdots i_l)$ to be isomorphic.

Proposition 3.2. *Two hom-Lie algebras $(mL, \Psi_\alpha(i, j))$ and $(mL, \Psi_\beta(k, l))$ are isomorphic if and only if α and β are quasi-conjugate automorphisms of L.*

Proof. First let α and β are quasi-conjugate automorphisms of L, i.e., there exist automorphisms ϕ and φ of L such that $\varphi \circ \alpha = \beta \circ \phi$. We define $\Phi_{(\phi,\varphi)}(i \leftrightarrow k, j \leftrightarrow l)$ to be the automorphism of mL that exchanges the i, k (respectively j, l)th entries by ϕ (respectively φ) and fixes other components, that is,

$$\begin{aligned} &\Phi_{(\phi,\varphi)}(i \leftrightarrow k, j \leftrightarrow l) X_{\cdots i \cdots k \cdots j \cdots l \cdots}(\cdots x \cdots y \cdots u \cdots v \cdots) \\ &\quad = X_{\cdots i \cdots k \cdots j \cdots l \cdots}\big(\cdots \phi(y) \cdots \phi(x) \cdots \varphi(v) \cdots \varphi(u) \cdots\big). \end{aligned} \tag{10}$$

It is obvious that $\Phi_{(\phi,\varphi)}(i \leftrightarrow k, j \leftrightarrow l)$ is indeed an automorphism of mL. By the definition of $\Psi_\alpha(i,j)$, $\Psi_\beta(i,j)$ and $\Phi_{(\phi,\varphi)}(i \leftrightarrow k, j \leftrightarrow l)$, it is easy to see that the following diagram

$$\begin{tikzcd} mL \arrow[r, "\Psi_\alpha(i,j)"] \arrow[d, "\Phi_{(\phi,\varphi)}(i\leftrightarrow k, j\leftrightarrow l)"'] & mL \arrow[d, "\Phi_{(\phi,\varphi)}(i\leftrightarrow k, j\leftrightarrow l)"] \\ mL \arrow[r, "\Psi_\beta(k,l)"] & mL \end{tikzcd}$$

commutes. Therefore by Definition 1.2, the hom-Lie algebras $(mL, \Psi_\alpha(i,j))$ and $(mL, \Psi_\beta(k,l))$ are isomorphic.

If the hom-Lie algebras $(mL, \Psi_\alpha(i,j))$ and $(mL, \Psi_\beta(k,l))$ are isomorphic, then there exists an automorphism Θ of mL such that the diagram

$$\begin{tikzcd} mL \arrow[r, "\Psi_\alpha(i,j)"] \arrow[d, "\Theta"'] & mL \arrow[d, "\Theta"] \\ mL \arrow[r, "\Psi_\beta(k,l)"] & mL \end{tikzcd}$$

commutes. Restricting Θ to the ideal L_i of mL, we have

$$\Theta\big(X_j\big(\alpha(L)\big)\big) = \Theta \circ \Psi_\alpha(i,j)(L_i) = \Psi_\beta(k,l) \circ \Theta(L_i). \tag{11}$$

Since α is an automorphism of L and Θ is an automorphism of mL, the fact that L_i is a simple ideal implies that $\Theta(X_j(\alpha(L))) = \Theta \circ \Psi_\alpha(i,j)(L_i)$ is also a simple ideal isomorphic to L. Hence $\Psi_\beta(k,l) \circ \Theta(L_i)$ is a simple ideal isomorphic to L. Equality (11) means that there are some automorphisms ϕ and φ of L such that $\varphi \circ \alpha = \beta \circ \phi$, and hence α and β are quasi-conjugate automorphism of L. □

Proposition 3.3. *A hom-Lie algebra of type $(mL, \Psi_\alpha(i,k))$ and a hom-Lie algebra of type $(mL, \Psi_\beta(j, j_1 j_2 \cdots j_l))$ are isomorphic if and only if α and β are quasi-conjugate automorphisms of L.*

Proof. Let α and β be quasi-conjugate automorphisms of L, i.e., there exist some automorphisms φ and ϕ of L such that $\varphi \circ \alpha = \beta \circ \phi$. Define $\Phi : mL \mapsto mL$ as follows: $\Phi(X_i(x)) = X_j(\phi(x))$, $\Phi(X_k(y)) = X_{j_1 j_2 \cdots j_l}(\varphi(y))$ and $\Phi(X_p(z)) = X_p(y)$ for $p \neq i, l$. It is easy to verify that Φ is an automorphism of mL that makes the diagram

$$\begin{tikzcd} mL \arrow[r, "\Psi_\alpha(i,k)"] \arrow[d, "\Phi"'] & mL \arrow[d, "\Phi"] \\ mL \arrow[r, "\Psi_\beta(j, j_1 j_2 \cdots j_l)"] & mL \end{tikzcd}$$

commute. So the hom-Lie algebras $(mL, \Psi_\alpha(i,k))$ and $(mL, \Psi_\beta(j, j_1 j_2 \cdots j_l))$ are isomorphic.

Now suppose $(mL, \Psi_\alpha(i,k))$ is isomorphic to $(mL, \Psi_\beta(j, j_1 j_2 \cdots j_l))$, then there will be an automorphism Θ of mL such that the diagram

$$\begin{tikzcd} mL \arrow[r, "\Psi_\alpha(i,k)"] \arrow[d, "\Theta"'] & mL \arrow[d, "\Theta"] \\ mL \arrow[r, "\Psi_\beta(j, j_1 j_2 \cdots j_l)"] & mL \end{tikzcd}$$

commutes.

In the diagram above, we have $\Psi_\alpha(i,k)(X_i(x)) = X_k(\alpha(x))$ for any $x \in L$, and hence $\Psi_\alpha(i,k)(L_i) \cong L_j$ is a simple ideal of mL. The image $\Theta \circ \Psi_\alpha(i,k)(L_i)$ of $\Psi_\alpha(i,k)(L_i) \cong L_j$ under the automorphism Θ is also a simple ideal of mL, which is isomorphic to $L_{j_1 j_1 \cdots j_l}$. The commutativity of the diagram gives us that $\Psi_\beta(j, j_1 j_2 \cdots j_l) \circ \Theta(L_i)$ is also a simple idea isomorphic to $L_{j_1 j_1 \cdots j_l}$. Therefore the equality

$$\Theta \circ \Psi_\alpha(i,k)\big(X_i(x)\big) = \Psi_\beta(j, j_1 j_2 \cdots j_l) \circ \Theta\big(X_i(x)\big)$$

means that for $\forall x \in L$,

$$X_{j_1 j_2 \cdots j_l}\big(\varphi \circ \alpha(x), \varphi \circ \alpha(x), \ldots, \varphi \circ \alpha(x)\big) = X_{j_1 j_2 \cdots j_l}\big(\beta \circ \phi(x), \beta \circ \phi(x), \ldots, \beta \circ \phi(x)\big)$$

holds for some automorphisms φ and ϕ of L. Therefore α and β are quasi-conjugate automorphisms of L. □

Proposition 3.4. *The hom-Lie algebra $(mL, \Psi_\alpha(i, i_1 i_2 \cdots i_k))$ is isomorphic to $(mL, \Psi_\beta(j, j_1 j_2 \cdots j_l))$ if and only if the automorphism α and β of L are quasi-conjugate.*

Proof. By Proposition 3.3, the hom-Lie algebra $(mL, \Psi_\alpha(i, i_1 i_2 \cdots i_k))$ is isomorphic to $(mL, \Psi_\alpha(i, p))$ and the hom-Lie algebra $(mL, \Psi_\beta(j, j_1 j_2 \cdots j_l))$ is isomorphic to $(mL, \Psi_\beta(j, q))$. According to Proposition 3.2, $(mL, \Psi_\alpha(i, p))$ is isomorphic to $(mL, \Psi_\beta(j, q))$ if and only if α and β are quasi-conjugate, So we arrive at our conclusion. □

Combining Propositions 3.2–3.4 we get the following theorem, which gives the isomorphic classes of non-trivial hom-Lie algebras on the semi-simple Lie algebra $mL = L \oplus L \oplus \cdots \oplus L$.

Theorem 3.1. *Let Π be the set of the quasi-conjugate classes of automorphisms of the simple Lie algebra L, then the isomorphic classes of hom-Lie algebras on mL $(m \geqslant 2)$ are $\{(mL, \Psi_\alpha(1,2)) \mid \alpha \in \Pi\}$.*

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